Historical Notes: Why the Quadratic Equation has Only One Root (according to abbaco masters)

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Ever since the first treatise on algebra by al-Khwārizmī (c. 800) it was recognised that certain quadratic problems could lead to double solutions. Early Arabic algebra used six canonical rules for solving problems, one for linear problems and five for quadratic problems. These rules were numbered 1 to 6. In order to avoid negative terms, quadratic problems with three terms were solved by three separate rules: (4) squares and roots equal number, (5) squares and number equal roots and (6) roots and number equal squares, corresponding with the modern-day equations: $ax^2 + bx = c$, $ax^2 + c = bx$ and $bx + c = ax^2$. A negative term within a polynomial expression was considered a defect which had to be 'restored' [1] (in Arabic al-jabr الجبر from which the name 'algebra' is derived). Negative solutions to quadratic problems were never considered, not to mention imaginary roots. However, for the fifth case, squares and number equal roots, al-Khwārizmī describes a rule which can lead to two positive solutions. He gives the example of $x^2 + 21 = 10x$ which leads to the solutions 3 and 7. Through three Latin translations of al-Khwārizmī's treatise in the twelfth and thirteenth centuries, double solutions to certain type of quadratic problems became known in Europe. Also Fibonacci comes to double solutions in the fifteenth chapter of his *Liber Abbaci* (1228). However, during the abbaco period, a tradition of mathematical practice between 1300 and 1500 in Northern Italy, the Provençe and Catalan regions [2], double solutions to quadratic problems silently disappear. For an answer to the curious question why this is the case we have to look at the specific rhetorical way of solving algebraic problems by abbaco masters.

There exist about 250 extant manuscripts of the abbaco period, mostly preserved in Italian libraries. About one third of them deal with algebra. All treatises on algebra follow a strict, repetitive and almost formalised structure for solving problems. After an enunciation of the problem (1), the solution always starts with a hypothetical reformulation of the problem text by use of an unknown, called a *cosa* or thing (2). Then, by manipulating some polynomial expressions one arrives at an equation for which a standard rule applies (3). This rule is applied to the problem values, such as the extraction of the root for quadratic problems (4). In an optional final step, the arrived solution is used in the problem enunciation to verify that it leads to the given values (5). Let us look at an example from the earliest treatise on algebra, on a problem of dividing 10 into two parts given that their product equals 20 [3, p. 313]:

1. And I want to say thus, make two parts of 10 for me, so that when the larger is multiplied against the smaller, it shall make 20. I ask how much each part will be.
2. Do thus, posit that the smaller part was a thing.
3. Hence the larger will be the remainder until 10, which will be 10 less a thing. Next one shall multiply the smaller, which is a thing, by the larger, which is 10 less a thing. And we say that it will make 20. And therefore multiply a thing times 10 less a thing. It makes 10 things less one *censo*, which multiplication is equal to 20. Restore each part, that is, you shall join one *censo* to each part, and you will get that 10 things are equal to one *censo* and 20 numbers.
4. Bring it to one *censo*, and then halve the things, from which 5 results. Multiply by itself, it makes 25. Remove from it the number, which is 20, 5 remains, of which seize the root, which it is manifest that it does not have precisely. Hence the thing is 5, that is, the halving less root of five. And we posited that the part, that is, the smaller, was a thing. Hence it is 5 less root of 5.

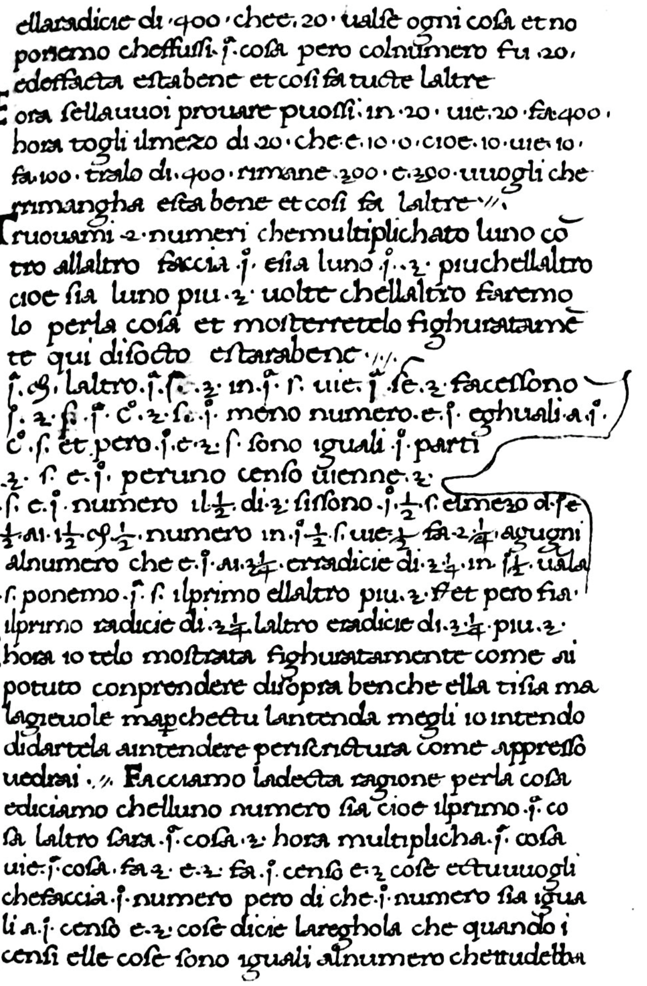

Figure 1: A similar problem from an early 15th century manuscript solved in an early form of symbolism (Used with permission, © British Library Board, Add 10363, f. 60r)

In modern symbolism the problem can thus be defined by two conditions

$$\begin{cases} a+b &= 10 \\ a.b &= 20 \end{cases}$$

in which the x of the quadratic equation takes the place of the smaller number a. Then $b = 10 - x$ or $x(10-x) = 20$. This leads to an equation of the fifth Arabic type: $10x = x^2 + 20$. In abbaco algebra equations are always normalised by dividing by the coefficient of the square term ('bring it to one *censo*'). As

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can be seen, the values a and b correspond with the two roots of this quadratic equation

$$x_1, x_2 = \frac{10}{2} \pm \sqrt{\left(\frac{10}{2}\right)^2 - 20}.$$

However, while the author knows that this equation can have two positive solutions he only subtracts the square root. The reason for this is that the *cosa* or x was chosen for the smaller number ('posit that the smaller part was a thing'). One particular value of the problem is thus represented by the unknown. The unknown can therefore not be considered an indeterminate value as in later algebra; it is an abstract representation for one specific quantity of the problem. Given that this recurring rhetoric structure, which is so important for the abacus tradition, commences by posing one specific value, it makes no sense to end up with two values for the unknown. If one starts an argumentation that the *cosa* represents the smaller part, one does not expect to end up with the value of the larger part. The concept of an unknown in the abacus tradition is so closely connected with a rhetorical structure that the choice of the unknown excludes double solutions by definition. Through the emergence of symbolism in the fifteenth century (as shown in the middle section of Figure 1) algebra became liberated from this rhetorical constraint and developed into a more abstract way of dealing with equations.

References

1 Heeffer, A. (2008) A conceptual analysis of early Arabic algebra, in S. Rahman, T. Street and H. Tahiri (eds.) *The Unity of Science in the Arabic Tradition: Science, Logic, Epistemology and their Interactions*, Kluwer Academic Publishers, Dordrecht, pp. 89–128.

2 Lawrence, S. (2009) Provençal mathematics, *Mathematics Today*, vol. 45, no. 6, pp. 219–220.

3 Høyrup, J. (2007) *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*, (Science Networks. Historical Studies, 34), Birkhäuser, Basel.